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Deriving average soliton equations with a perturbative method

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The method of multiple scales is applied to periodically amplified, lossy media described by either the nonlinear Schrödinger (NLS) equation or the Korteweg–de Vries (KdV) equation. An existing result for the NLS equation, derived in the context of nonlinear optical communications, is confirmed. The method is then applied to the KdV equation and the result is confirmed numerically.

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Many physical systems can be described with integrable nonlinear evolution equations [1]. The two most prolific—as measured by number of applications—are the nonlinear Schrödinger (NLS) equation and the Korteweg–de Vries (KdV) equation (see Ref. [2], for example). Each of these equations is applicable to systems which are both weakly nonlinear and weakly dispersive.

The concept of periodic amplification of the NLS equation has recently been investigated with regard to long-haul optical communication systems using the *average or guiding soliton* approach [3–6]. Periodic amplification of the KdV equation has not yet been addressed, but is potentially applicable to a new electrical oscillator [7]. This oscillator consists of a loop of lumped-element nonlinear transmission line with a single amplifier to compensate for losses. In Ref. [7], the oscillation modes were compared with the periodic and soliton solutions of the Toda lattice, but recent experiments have confirmed that the KdV equation is a more appropriate description. Hence the oscillator may be considered a periodically amplified KdV system, and an average soliton approach is likely to be valuable.

This paper considers the periodic amplification of both KdV and NLS systems, in such a way that the basic perturbative nature of the problem is apparent. The NLS case is revisited, using the *Method of Multiple Scales* (MMS), before the KdV problem is approached with the same technique. Naturally, the *soliton* is a salient feature in both systems.

A typical optical fiber communication link consists of an arrangement of glass fiber divided into equal sections by optical amplifiers; a schematic is shown in Fig. 1. Here, L is the normalized distance separating amplifiers of gain μ, and $u_0(t)$ is the input to the system. The propagation between amplifiers—located at $z=L$, $2L$, $3L, \ldots,$ — is described by a NLS equation,

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial t^2}+|u|^2 u=-i\Gamma u \ , \tag{1}$$

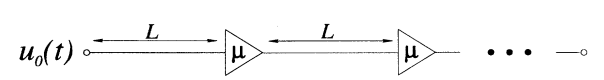

FIG. 1. Periodically amplified nonlinear transmission system.

with a normalized loss coefficient Γ [8]. From a communications perspective, the propagation of individual pulses is of chief concern. It has been shown that pulses can suffer large attenuations between amplifiers, yet still behave like solitons of the homogeneous, lossless medium $(\Gamma=0,\mu=1)$ [3–6].

Mollenauer, Evangelides, and Haus [3] have given a conceptually simple derivation of this effect by exploiting the fact that the pulse shape does not change appreciably between amplifiers. They show that the *global* behavior of the system can be described with a homogeneous NLS equation,

$$i\frac{\partial \hat{u}}{\partial z}+\frac{1}{2}\frac{\partial^2 \hat{u}}{\partial t^2}+\rho|\hat{u}|^2\hat{u}=0 \ , \tag{2}$$

where $\hat{u}(t,z)$ is the wave form immediately following each amplifier and $\rho=[1-\exp(-2\Gamma L)]/(2\Gamma L)$ is a factor to account for the attenuation between amplifiers. The single soliton solution to Eq. (2) can propagate—without serious distortion—over large distances and many amplifier spans [3,4,6]. We present a more formal derivation of Eq. (2), which will later be applied to periodically amplified KdV systems.

If the distance between amplifiers is much less than that over which significant nonlinear and dispersive effects can occur, then these effects can be treated as perturbations [3]. Conversely, the attenuation is typically very large: pulse power may change by a factor of 10 between amplifiers. Mollenauer, Evangelides, and Haus [3] take the approach that the net effect is the approximate preservation of the pulse *shape* — here, we consider the perturbation problem directly.

The perturbative nature of the nonlinearity, and dominance of the loss, can be made explicit by introducing a small quantity, $\epsilon \ll 1$ and substituting $z=\epsilon z'$ and $\Gamma=\Gamma'/\epsilon$ into Eq. (1), so that

$$i\frac{\partial u}{\partial z'}+\epsilon\left[\frac{1}{2}\frac{\partial^2 u}{\partial t^2}+|u|^2 u\right]=-i\Gamma' u \ . \tag{3}$$

Here, z is considered a "long" length scale and z' as a "short" length scale [9]. The fact that the problem has been phrased as a perturbation problem, with two natural scales, suggests that the MMS may be helpful [9,10]. The MMS assumes that the solution can be expressed as a

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power series in ϵ— with z and z' treated as *independent* variables, so that

$$u(t,z)=u_0(t,z,z')+\epsilon u_1(t,z,z')+\epsilon^2 u_2(t,z,z')+\cdots \quad (4)$$

This equation can be substituted into Eq. (3) and terms collected at different orders of ϵ. Collecting terms at $O(\epsilon^0)$, we have

$$\frac{\partial u_0}{\partial z'}=-\Gamma' u_0 , \quad (5)$$

which can be solved to find

$$u_0=\hat{u}(t,z)\exp(-\Gamma' z') , \quad (6)$$

where $\hat{u}$ is an (as yet) arbitrary function of z and t— but not of z'. This is almost the solution Mollenauer, Evangelides, and Haus assumed between amplifiers; the authors remark that $\hat{u}$ varies with z on a large scale, but, here, this fact is accounted for in the notation. The following analysis shows that $\hat{u}(t,z)$ is indeed the solution to Eq. (2). At $O(\epsilon^1)$, we have

$$-i\frac{\partial u_1}{\partial z'}-i\Gamma' u_1=i\frac{\partial u_0}{\partial z}+\frac{1}{2}\frac{\partial^2 u_0}{\partial t^2}+|u_0|^2 u_0 . \quad (7)$$

The MMS enables an equation for $\hat{u}(t,z)$ to be found by requiring that the perturbation series (4) is valid for $z'=O(\epsilon^{-1})$. Since it is clear that the solution $u(t,z)$ tends to zero—we have not yet considered the amplification—then $\{u_0,u_1,\ldots,\}$ must also tend to zero. Therefore, so that u_1 tends to zero, we substitute Eq. (6) into the right-hand side of Eq. (7) and set it to zero, which yields

$$-i\frac{\partial \hat{u}}{\partial z}=\frac{1}{2}\frac{\partial^2 \hat{u}}{\partial t^2}+\exp(-2\Gamma' z')|\hat{u}|^2\hat{u} . \quad (8)$$

The final step is to average Eq. (8) over one amplification period—with respect to the short scale z'— and, therefore, remove the dependence on z', i.e.,

$$\frac{1}{L/\epsilon}\int_0^{z'=L/\epsilon}\left[i\frac{\partial \hat{u}}{\partial z}+\frac{1}{2}\frac{\partial^2 \hat{u}}{\partial t^2}+\exp(-2\Gamma' z')|\hat{u}|^2\hat{u}\right]dz' =0 . \quad (9)$$

When this integral is evaluated, we reach the result shown in Eq. (2). The average is valid when the gain is precisely sufficient to compensate for the attenuation between amplifiers—i.e., from Eq. (6), $\mu=\exp(\Gamma L)$. Mollenauer, Evangelides, and Haus give Eq. (8) as

$$-i\frac{\partial \hat{u}}{\partial z}=\frac{1}{2}\frac{\partial^2 \hat{u}}{\partial t^2}+\exp(-2\Gamma z)|\hat{u}|^2\hat{u} . \quad (10)$$

To obtain Eq. (2), this equation must also be integrated over one amplifier span. To do this, the authors place functions of $\hat{u}$ outside the integral of the right-hand side of Eq. (10) ($\hat{u}$ does not vary significantly between amplifiers) *but* integrate the left-hand side (another function of $\hat{u}$) to obtain a differential term, $\Delta\hat{u}$. This difficulty is avoided in Eq. (9) because the characteristic length scales have been separated from the outset.

The key element in this development is that the distance between amplifiers is such that the pulse shaping effects of dispersion and nonlinearity can be treated as perturbations. Another famous equation which incorporates the effects of both dispersion and nonlinearity—and supports solitons—is the Koretweg–de Vries (KdV) equation [11]. To draw an interesting parallel between two solitonic systems, and to introduce the possibility of periodically amplified KdV systems, we show that the lossy KdV equation can be treated in the same manner as the NLS equation.

The system under consideration is that shown in Fig. 1, except that the propagation between amplifiers is described by the damped KdV equation,

$$\frac{\partial u}{\partial z}+6u\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial t^3}=-\Gamma u . \quad (11)$$

A key difference between the NLS and KdV equations is that the speed of propagation in the latter is amplitude dependent—and hence will change as a pulse attenuates between amplifiers. We apply the MMS method to this system, to deduce an equation which describes the overall system behavior. To make the nonlinear and dispersive effects apparent as perturbations, we again set $z=\epsilon z'$ and $\Gamma=\Gamma'/\epsilon$ and proceed *exactly* as before, which leads to

$$\frac{\partial \hat{u}}{\partial z}+6\eta\hat{u}\frac{\partial \hat{u}}{\partial t}+\frac{\partial^3 \hat{u}}{\partial t^3}=0 \quad (12)$$

and where $\eta=[1-\exp(-\Gamma L)]/(\Gamma L)$. Equation (12) plays the same role as Eq. (2) did previously—it describes a pulse immediately following each amplifier.

Equation (12) can readily be tested by numerically comparing its behavior to that of the actual system (Fig. 1). A straightforward approach is to assume that any wave forms in Fig. 1 can be expressed with a t-periodic Fourier series—nonperiodic cases can be catered for by considering a sufficiently large period. This leads to an infinite set of ordinary differential equations (ODE) which can be truncated and solved with a standard ODE solver [12]. By choosing suitable input wave forms, $u_0(t)$, Eq. (12) can be trialed against the actual behavior of the system. To this end, we set the system details to be $L=0.1$, $\Gamma=10\ln 10\approx 23.0$, $\mu=\exp(\Gamma L)=10$, and $\eta^{-1}\approx 2.56$.

The single soliton solution to Eq. (12) is given by

$$\hat{u}(t,z)=a\eta^{-1}\mathrm{sech}^2\frac{\sqrt{a}}{2}(t-2az) . \quad (13)$$

An initial wave form corresponding to this solution ($a=\eta$) is $u_0(t)=\hat{u}(t,z=0)=\mathrm{sech}^2((\eta/2)^{1/2}t)$. Figure 2 shows the evolution of this wave form when launched into the system in Fig. 1 (solid lines). Each pair of solid lines correspond to wave forms at the beginning and end of individual sections of the lossy KdV medium, after 0, 120, and 240 amplifiers have been passed. The solution given by Eq. (13)—which is the solution to the average soliton equation, equation (12)—is plotted, at discrete points, with open circles. Clearly the solution to the average KdV equation coincides with the simulated wave forms which follow each amplifier, just as for the average NLS equation. In fact, a stronger result can be shown by considering another input wave form.

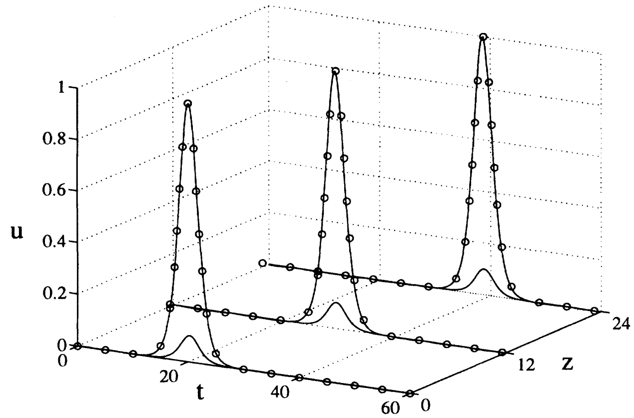

FIG. 2. Evolution of a single, periodically amplified, pulse: gain, $\mu=10$, amplifier spacing, $L=0.1$. Each pair of solid lines correspond to wave forms at the beginning and end of a single amplification period L, as shown in Fig. 1, after 0, 120, and 240 amplifiers have been passed. Open circles correspond to the soliton solution of the average soliton equation, Eq. (12).

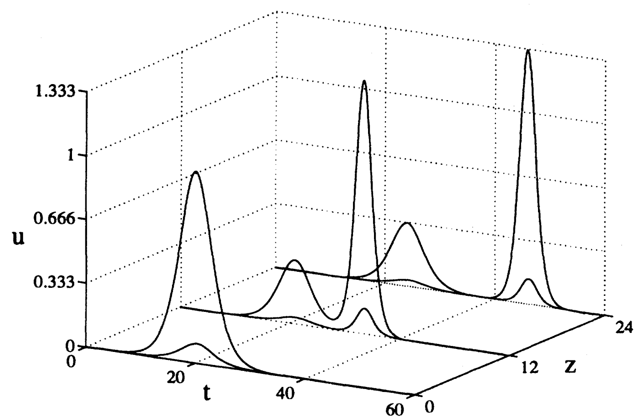

FIG. 3. Evolution of two, periodically amplified, pulses. System parameters are the same as for Fig. 2. Each pair of lines correspond to wave forms at the beginning and end of a single amplification period, L, after 0, 120, and 240 amplifiers have been passed. The amplitudes of the individual pulses agree with the values predicted from inverse scattering theory, as applied to the average soliton equation (12).

By applying inverse scattering theory to Eq. (12), it can be shown that $u_0(t)=\hat{u}(t,z=0)=\text{sech}^2((\eta/6)^{1/2}t)$ develops into exactly two solitons—of the form given in Eq. (13)—with amplitudes $\frac{1}{3}$ and $\frac{4}{3}$ [13]. Figure 3 shows the development corresponding to this initial condition when launched into the periodically amplified KdV system; clearly two solitons with correct amplitudes develop. Again the solid pairs of lines correspond to wave forms at the beginning and end of individual sections of the lossy KdV medium, after 0, 120, and 240 amplifiers have been passed. This result shows that Eq. (12) is not restricted to predicting the behavior of single pulses. A similar result has been shown recently, in the optical regime, where second-order average solitons have been observed [14].

When these simulations are repeated with a much larger distance between amplifiers, say $L=1$, Eq. (12) can no longer be used to predict the global behavior of the system. This is because the perturbation method becomes invalid when the nonlinearity and dispersion cannot be considered as perturbations over an amplifier span, L. In the optical regime the soliton period is a guide to how small L must be [3] and Kelly [15] has shown how an instability can develop if L is too large. We have not determined a quantity corresponding to the soliton period for the KdV case, but Fig. 3 gives a reasonable indication of the lengths over which nonlinearity and dispersion can have an appreciable effect.

In conclusion, the MMS has been used to formalize the result obtained by Mollenauer, Evangelides, and Haus [3]. That nonlinearity and dispersion can be balanced, on average, for the periodically amplified NLS equation is a surprising result—perhaps more so for the KdV case. Whereas both nonlinearity and dispersion effect a NLS pulse only on a large scale, the *speed* of a KdV pulse is directly related to its amplitude. Thus, there is an attribute of the KdV pulse which can *vary significantly* over one amplification period. Heuristically speaking, an average description is still possible because the pulse travels faster at the beginning of the amplification period and slower near the end, resulting in the correct *average* speed. If this fact is accepted *a priori* then Mollenauer, Evangelides, and Haus argument can be equally well applied to the periodically amplified KdV system by disregarding the dependence of the pulse's speed on its amplitude.

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